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Invading clusters in fractal media

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Abstract. Two-dimensional invasion percolation simulations into regular fractals are performed to study the dependence of the invading cluster fractal dimension on the geometry of the medium. The fractal dimension of the invading cluster D_i neither depends on the average coordination number of the network, nor on the links of the backbone. We find that D_i only depends on the fractal dimension of the backbone of the medium D_{bb} and varies linearly with it.

Among the most important models for simulating fluid displacement in porous media we can mention the Eden (1961), diffusion limited aggregation (Witten and Sander 1983) and invasion percolation (Wilkinson and Willemsen 1983) models. The diffusion limited aggregation (DLA) model is used for emulating systems in the viscous regime. The other two refer to the capillary regime. Using the DLA model and invasion percolation, fractal growth patterns are observed on an Euclidean support. Nevertheless, these models can be applied directly on media which are not necessarily Euclidean. This fact makes these models more interesting since it has been shown that the surface of the pores can be characterized with a fractal dimension (see Katz and Thompson (1985), Hansen and Skjeltorp (1988), Thompson *et al* 1987). Furthermore, Oxaal *et al* in 1987 showed that for both, the DLA and the Eden models, fluid displacement in a medium depends very strongly on its fractality.

The invasion percolation model represents a more realistic description of fluid displacement in a porous medium than the Eden model, since the medium is a pore structure, with pores of vastly varying sizes. The purpose of this work is to study the dependence of the fractal dimension that the invading fluid acquires on the geometry of the porous medium which we represent as a fractal network. In this paper we present some important results of this study, where we have used the model of site invasion percolation to examine the dependence on average coordination number, links structure and the dimension of the percolating backbone.

Using invasion percolation, Paredes and Octavio (1990) found that the fractal dimension of the invading fluid (D_i) tends to be the same as the fractal dimension of the porous medium ($D_f = 1.89$), when simulating invasion in a spanning cluster saturated with an infinitely compressible fluid, at the critical probability p_c. The correspondence between the above two structures is straightforward in this case, which corresponds to invasion percolation without trapping. When the original fluid is infinitely incompressible (invasion percolation with trapping) Paredes and Octavio

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(1992a) found the following results: (i) $D_i = 1.37$ when invading the spanning cluster at p_c; (ii) $D_i = 1.82$ when invading the spanning cluster $p \gg p_c$. In this case the geometric correspondence between the structure of the network and that of the invading fluid is not straightforward. Our objective is to find the dependence of D_i on D_f and to determine which are the relevant parameters that influence such a dependence.

The invasion percolation model introduced by Wilkinson and Willemsen is used to describe immiscible bi-phasic fluid displacement into porous media, so that when injecting a fluid from one side of the network, it percolates to the opposite side, pushing out the original fluid of the medium. This model only takes into account the case where, the capillary number is much smaller than 1. That is, the capillary forces dominate viscous ones, so that the fluid displacement is due exclusively to capillary phenomena. In our work we consider only drainage, which means that the invading fluid is the non-wetting phase. In this model the invading fluid, in order to continue the invasion, will always choose the largest pore radius that the interface sees at a given instant, since, the fluid displacement takes place pore by pore.

A fractal medium is conformed by dead-ends, links and blobs (Stanley 1977). The dead-ends are paths where there is no transport (fluid, current, etc). For invasion percolation with trapping we have that the dead-ends are always trapping zones since, when the interface is localized at the beginning of the zone, the invading fluid cannot continue penetrating it, because the original fluid in that region has no way out. This leads us to state that an invaded network with or without dead-ends are equivalent i.e. in both cases the invading fluid has exactly the same structure. We can then say that in some sense the invading fluid only depends on geometry of the network without dead-ends. The network without dead-ends is called the backbone, which is conformed by links and blobs. The links are those sites of the network that are singly connected, that is, when we take out a link from our medium, the connection between opposite sides is interrupted. The blobs are the multiply connected sites. When we subtract a site from the blob, there will always exist another option in the walking path. In a fractal, it is important to remember that the fractal dimension of the blobs (D_b) tends to be the same as the fractal dimension of the backbone (D_{bb}), since as the length tends to infinity the blobs clearly dominate over the links (Pike and Stanley 1981). We have studied the dependence of the fractal dimension of the invading cluster on the backbone and links, as well as on the average coordination number of first occupied neighbours z.

We use regular fractal networks to represent porous media. The fractal networks are regular self-similar structures. The regular fractals employed in this work were constructed using the following procedure. The initial structure, $k = 0$, is a square of length $L(0)$ with $N(0)$ occupied sites. In the following step or second generation, $k = 1$, each one of the occupied sites is substituted by the $k = 0$ structures but rotated by a preselected angle $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi\}$ obtaining in this form a network of length $L(1) = L(0)^2$ with $N(1) = N(0)^2$ occupied sites. The procedure is continued with the same sequence of rotations, generating a fractal at the scale wished. Figure 1 shows a fractal generated as described previously taking $\theta = 0$ for all occupied sites. The fractal dimension of this object is determined by the relation,

$$D_f = \frac{\log(N(k))}{\log(L(k))}. \tag{1}$$

This definition of the fractal dimension could also be used to compute, for example, the fractal dimension of the links (D_l) or the fractal dimension of the invading fluid

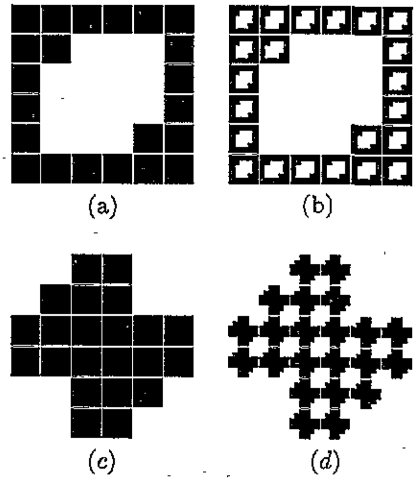

Figure 1. (*a*) The $k=0$ initial structure. The black sites identify the occupied sites. (*b*) The $k=1$ structure which represent the second scale of generation of the object that appear in (*a*). $z=3.0$ and $D_f=1.73$, when $k\to\infty$. (*c*) The $k=0$ initial structure of an array with the same number of occupied sites of the array that appear in (*a*). Notice that the occupied sites are distributed differently. (*d*) The $k=1$ structure which represent the second scale of generation of the object that appear in (*c*). $z=3.2$ and $D_f=1.73$, when $k\to\infty$.

(D_i), introducing into the equation the number of links N_l and the number of sites invaded N_i, respectively. For these cases the fractal dimension has the correct value when k tends to infinity.

On the other hand, the average coordination number of the fractal network is defined by,

$$z=\lim_{k\to\infty} z(k)=\lim_{k\to\infty}\frac{\sum_{i=1}^{N(k)} z_i}{N(k)} \tag{2}$$

where z_i is the number of nearest neighbours of the occupied site i, and $N(k)$ the total number of occupied sites, for the k structure. We calculate $z(k)$ for several generations and then obtain the coordination number in the asymptotic limit when the difference of two consecutive values of $z(k)$ is less than a given tolerance.

We chose to work with regular fractals because the way of generating them allows control of their characteristic parameters, which is indispensable for a quantitative study. Furthermore, this allows for the generation of networks in a variety of dimensions rather than the two dimensions which can be obtained in standard percolation, 1.89 at p_c and 2 for $p\gg p_c$.

In invasion percolation a random number between 0 and 1 is generated and is assigned to each site on the network making this number and the radius of the pore correspond directly. Then, when invasion percolation simulations are performed in these networks invading clusters are generated. We used the box-counting method (Feder 1988) for measuring the different fractal dimensions. The simulations were carried out on SUN workstations. All networks used in the invasion were of size no less than 635×1270 and as large as 1296×2592. The different D_i values to which we will refer later correspond to the average of at least 20 realizations carried out for each network. The reported error bars for the averaged D's correspond to the standard deviation. The error bars are probably larger than the error bars reported because of the finite-size effects and systematic errors.

In order to study the dependence of D_i with respect to the average coordination number z, we constructed fractals of the same dimensions with neither dead-ends nor links. We generated these networks considering arrays of the same length in the $k=0$ initial structure. Every $k=0$ initial structure was occupied with the same number of sites; the occupied sites were differently distributed in each initial structure, in order to obtain networks with different z. In figure 1 we show two examples of the process mentioned above. In figure 2 we show a typical invasion process of our simulations. We show in figure 3, that for networks of different z, the invading fluid has the same fractal dimension, within the limits of the estimated error. Additionally, D_i is always 1.82, using the invasion percolation model on percolating clusters for $p \gg p_c$ (Paredes and Octavio 1992a), despite the fact that the average coordination number z changes as a function of p. From this, we have another confirmation that D_i does not depend on the coordination number of the porous media. But D_i cannot depend on z because the average coordination number is a parameter that describes the existing connectivity between the pores only at the smallest scales. This effect does not change the fractal dimension that the invading fluid acquires since the fractal dimension should characterize the system at all scales. Wilkinson and Willemsen found that when invading two Euclidean networks, one triangular ($z=6$) and the other square ($z=4$), D_i in the first

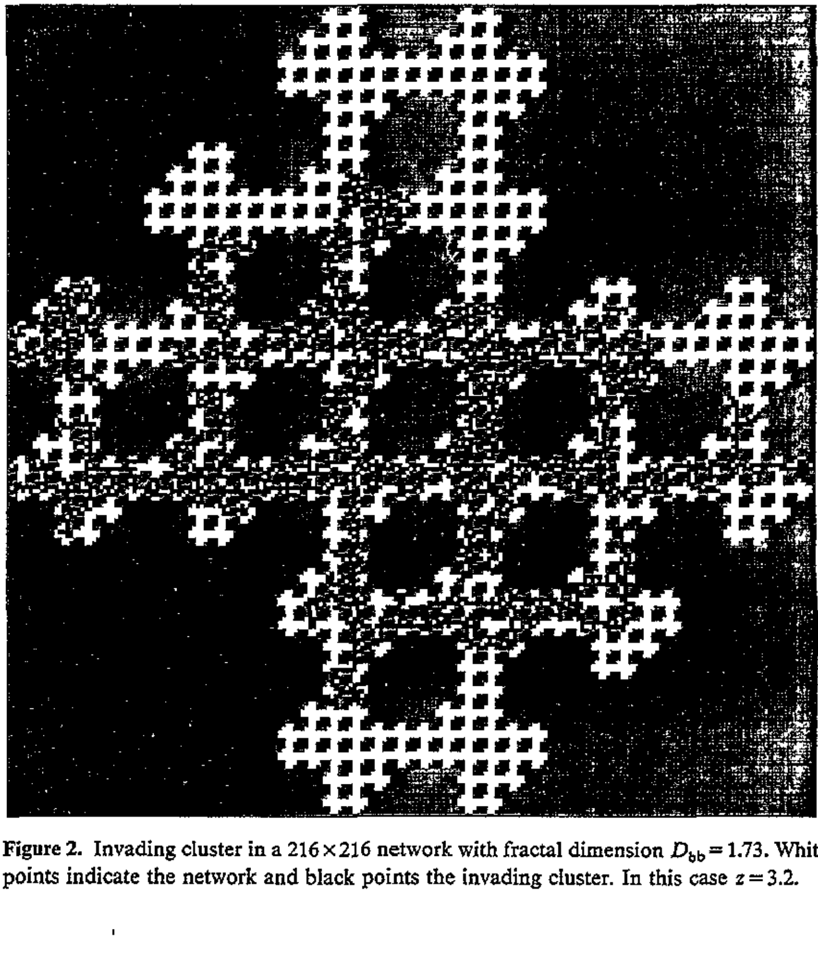

Figure 2. Invading cluster in a 216×216 network with fractal dimension $D_{bb}=1.73$. White points indicate the network and black points the invading cluster. In this case $z=3.2$.

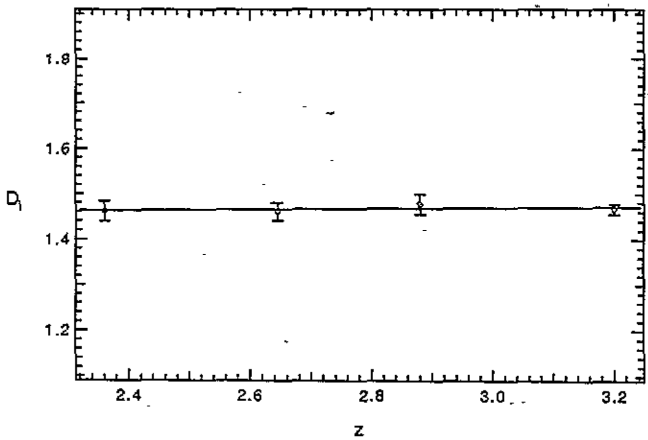

Figure 3. The fractal dimension of the invading cluster versus the average coordination number z. All the networks have the same fractal dimension $D_{bb}=1.73$ and do not have links. The fractal dimension of the invading cluster is independent of the average coordination number. The invasions were performed in lattice of size 1296×2592. The different values of D_i were the average of 20 simulations.

case was 1.88 ± 0.02 and in the second case D_i was equal to 1.82 ± 0.02. They attributed the difference between both values of D_i to the fact that in the first case p_c is equal to $0.5927\ldots$, and in the second case p_c is equal to 0.5. If invasion percolation with trapping is used, it is known that for Euclidean network of bonds (where $p_c=0.5$ and $z=6$) and for Euclidean network of sites ($p_c=0.5927\ldots$, and $z=4$) D_i is the same, within the estimated error (see Chandler *et al* 1982).

In fluid transport the necks (the sub-set composed by connected links) always plays a decisive role on the transport. On the other hand, in invasion percolation with trapping, when the invading fluid penetrates one of the necks, the uninvaded zones leading up to that neck remain definitely entrapped. In our work we experiment with networks that contain both links and blobs. All the networks have the same fractal dimension of the backbone but different structure of links. We generated these networks establishing a rule of rotation in the second generation scale in the way mentioned above (see figure 4), so that no dead-ends appear that would make D_f different from D_{bb}. Furthermore, the way we rotate the basis of the fractal in the second scale leads us to generate networks of different D_l. When we made our simulations, we found that D_i remains constant, within the estimated error. This fact can be clearly observed in figure 5. We conclude that D_i does not depend upon the structure of links. The above result arises from the fact, that the links only affect the structure of the invading fluid at the larger scales present in the invaded blob prior to the invasion of each link. All the remaining scales are not sensitive to the presence of links. On the other hand, D_i will never be smaller than D_m, where D_m is the fractal dimension of the minimum path that connects two arbitrary sites extremes on the backbone (Laidlaw *et al* 1987). One finds that when the size tends to infinity, the fractal dimension of the sites that belong to the blobs contained in the minimum path, dominate over the fractal dimension of the links. This confirms that D_i, even in the extreme case, is completely dominated by the fractal dimension of the sites contained in the blobs of the network and not by the fractal dimension of the links (Pike and Stanley 1981).

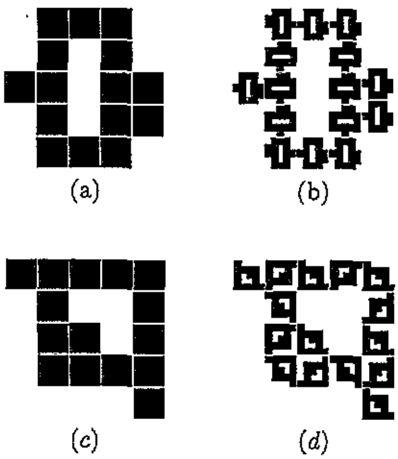

Figure 4. (a) The $k=0$ initial structure. The black sites identify the occupied sites. (b) The $k=1$ structure which represent the second scale of generation of the object that appear in (a). $D_l=0.0$ and $D_f=1.68$, when $k\to\infty$. (c) The $k=0$ initial structure of an array with the same number of occupied sites of the array that appear in (a). Notice that the occupied sites are distributed differently. (d) The $k=1$ structure which represent the second scale of generation of the object that appear in (c). $D_l=0.46$ and $D_f=1.73$, when $k\to\infty$. In (b) and (d), notice that we establish a rule of rotation starting on this second scale, so that no dead-ends, that would make D_f different from D_{bb}, appear.

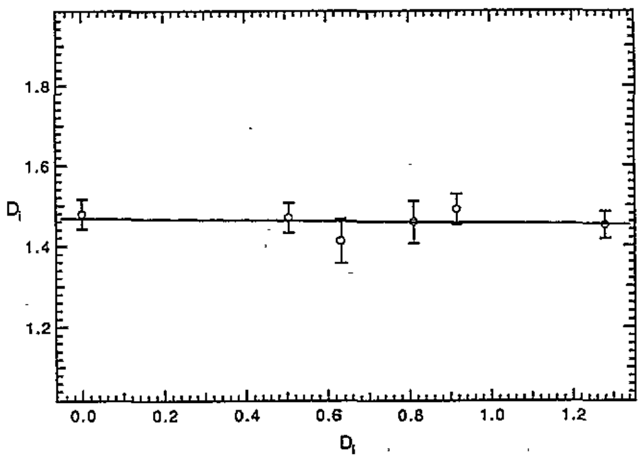

Figure 5. The fractal dimension of the invading cluster versus the dimension of the links D_l. All the networks have the same fractal dimension of the backbone, $D_{bb}=1.68$. The fractal dimension of the invading cluster is independent of the dimension of the links. The invasions were performed in lattice of size 625×1250. The different values of D_i were the average of 50 simulations.

The trivial case is when the network is formed only by the links. In this situation, it is easy to see that the minimum path is the network, and that the structure of the invading fluid is exactly the same as that of the network.

Finally, we generated networks of different fractal dimensions of the backbone (D_{bb}), composed only by blobs, and we simulate invasions on each network. In

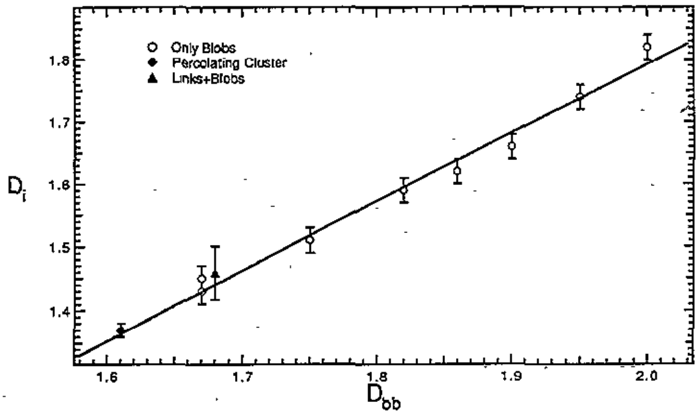

Figure 6. The fractal dimension of the invading cluster versus the dimension of the backbone. The links+blobs point come from the fit of the points in figure 5. The fractal dimension of the invading cluster depends on the fractal dimension of the backbone and this dependence is linear. The invasions performed in the networks composed only by blobs were of size 1296×2592. The different values of D_i were the average of 100 simulations.

figure 6, we note the surprising fact that D_i varies linearly with D_{bb}, within the estimated error, and the slope is approximately equal to one. Furthermore, we observe that the D_i (=1.37) obtained from a percolating cluster ($D_f = 1.89$ and $D_{bb} = 1.61$) falls on the same line. Additionally, if we fit the D_i data from figure 5, that point also falls on the same line. The errors bars of D_i corresponding to the networks of only blobs (shown in figure 6) are the errors that come about when the least squares fit is carried out in order to calculate the different values of D_i.

Furthermore, in figure 6 we show, as an example, two different supports with $D_{bb} = 1.67$ that give rise to different values of D_i. The difference between them could be the fractal dimension of the minimum path D_m (for the larger D_i the value of D_m is 1.05 and for the other point D_m is 1.00), difference which could be neglected for the other cases we studied. We suggest an extensive study of this possible dependence on other types of fractals; see Paredes and Octavio (1992b).

In conclusion, we have shown that for a variety of regular fractal networks, the only crucial parameter that determines the fractal dimension of the invading fluid in invasion percolation with trapping is the fractal dimension of the backbone of the medium. Furthermore, we showed a linear dependence between D_i and D_{bb}. Additionally, in this paper it was shown that the fractal dimension of the invading fluid neither depends on the average coordination number of the network, nor on the fractal dimension of the links.

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